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Stability of swirling flows with radius-dependent density

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The inviscid instability of heterogeneous swirling flows with radius-dependent density is investigated and secular relations for the instability growth rates for several different flow configurations are obtained from explicit solutions of the governing equations. It is found, in agreement with a sufficiency condition for the stability of such flows obtained earlier, that they are stable to both axisymmetric and non-axisymmetric infinitesimal modes whenever the density is a monotonic increasing function of radius and at the same time the radial variations in both the angular and axial velocity components remain small. The instability mechanisms present in these flows are both of centrifugal and of shear origin, the classical Rayleigh–Synge criterion being a condition for centrifugal stability. It is shown, via several counter examples, that the Rayleigh–Synge criterion for the stability of swirling flows is generally neither a necessary nor a sufficient condition when non-axisymmetric disturbances are considered or large shears exist in the flow. Very stable flows occur when the angular and axial velocity components have no radial variation and simultaneously the density increases with radius as is the case in a typical centrifuge.

1. Introduction

It is our purpose in this paper to examine the inviscid instability of heterogeneous swirling flows for infinitesimal disturbances of arbitrary asymmetry. An understanding of the hydrodynamic stability characteristics of such flows is of considerable practical interest not only for flows occurring in counterflow centrifuges but also for certain vortex stabilization schemes such as that proposed for the containment of fusion products from laser-imploded deuterium pellets. Although much is known concerning the stability of constant density swirling flows (Chandraskhar 1961; Howard & Gupta 1962; Michalke & Timme 1967), relatively little attention has been given to the stability of flows with radius-dependent density subjected to non-axisymmetric disturbances. The existing stability investigations of such heterogeneous swirling flows have concerned themselves mainly with the special profiles associated with rotating jets (Ponstein 1959; Alterman 1961) or with Rankine vortices with subregions of constant density and axial velocity (Uberoi, Chow & Narain 1972). Such flows are generally found to be unstable via a Kelvin–Helmholtz instability mechanism because of discontinuities in their steady-state velocity components. It was not until relatively recently that a general sufficiency criterion for the stability of heterogeneous swirling flows with continuous but arbitrary radial variation in the velocity components and density was obtained (Kurzweg 1969). The stability criterion found reduces to the familiar Rayleigh–Synge criterion for the special case of axisymmetric modes and no axial velocity gradients but at the same time suggests that flows may be unstable to non-axisymmetric disturbances even when the product of the density and the square of the circulation is a monotonic increasing function of radius. The existence of such non-axisymmetric modes for certain swirling flows is supported by the experimental observations of Weske & Rankin (1963) and of Johnston (1972) and also by some recent analytical results of Fung & Kurzweg (1973).

We wish here to present some exact solutions of the equations governing the stability of heterogeneous swirling flows both with continuous and with discontinuous radial density variation. Our objectives are to obtain an estimate of the accuracy of the sufficiency condition for these flows and to show that generally there are both centrifugal and shear instability mechanisms present in these flows. Large radial gradients in the axial and angular velocity components will be found to be destabilizing while large positive radial density gradients tend to stabilize the flow. In contrast to the results obtained by Howard & Gupta (1962) for constant density swirling flows, it will be shown, both from a re-examination of the sufficiency condition and from calculated values of the oscillation-amplification factor for specific flows, that such flows can be guaranteed to be stable for all infinitesimal modes of instability if the density is a monotonic increasing function of radius but the gradients in the angular and axial velocity components remain small. Such stable conditions are found, for example, in centrifuges provided that the counterflow current remains small.

2. Governing equations and interfacial conditions

We consider a heterogeneous swirling flow confined within an annular region $R_1 \leqslant r \leqslant R_2$ between two concentric cylinders. The flow has a radius-dependent steady-state velocity field $[0, r\Omega_0(r), W_0(r)]$ and density $\rho_0(r)$. These velocity components and density may be either a solution of the Navier–Stokes equation or simply chosen as convenient profiles satisfying continuity requirements. We shall confine ourselves here, for purposes of mathematical simplicity, to small amplitude disturbances and neglect both viscous and gravitational effects on the instability growth rates. Furthermore, the flow is assumed to be incompressible as would be the case for a radially stratified liquid.

Following the usual normal-mode analysis, we assume that the flow is disturbed by infinitesimal, three-dimensional, time-dependent perturbations whose non-radial dependence has the exponential form $\exp i(kz + m\theta - \omega t)$, where k is the axial wavenumber, m is the integer azimuthal wavenumber and ω is the complex oscillation-amplification factor. Substituting the sums of the steady-state solutions and perturbations into the governing Euler equation, continuity equation and incompressibility condition, one finds, within the framework of the above approximations, that the linear stability behaviour of the flow is governed by

$$\left.\begin{aligned} Dp+\rho_0[iNu-2\Omega_0 v]-\rho r\Omega_0^2 = 0, \quad p+\rho_0 rm^{-1}[Nv-iD^*(r\Omega_0)u] = 0, \\ p+\rho_0 k^{-1}[Nw-i(DW_0)u] = 0, \quad D^*u+ik(sv+w) = 0, \quad iN\rho+(D\rho_0)u = 0, \end{aligned}\right\} \quad (1)$$

where $D = d/dr$, $N = kW_0+m\Omega_0-\omega$, $D^* = d/dr+r^{-1}$ and $s = m/rk$ is the disturbance asymmetry parameter, which is zero for axisymmetric modes and infinite for azimuthally symmetric disturbances. Here u, v and w are the perturbations in the radial, azimuthal and axial velocity, p is the pressure perturbation and ρ the perturbation in density. We shall be concerned only with temporal instabilities, for which the wavenumbers will be real while $\omega = \omega_r+i\omega_i$ will usually be complex. Flow instability occurs whenever ω_i is positive, or more accurately, since the equations are invariant under complex conjugation, whenever ω_i is different from zero. The stability boundary is defined as the curve separating regions of neutral stability ($\omega_i = 0$) from regions of instability ($\omega_i > 0$).

Combining equations (1) by eliminating the variables v, w and ρ leads to the two first-order equations

$$(k^2+m^2/r^2)\,p = i\rho_0\{[k(DW_0)+mD^*(r\Omega_0)/r]\,u-N(D^*u)\} \quad (2)$$

and

$$NDp+(2m\Omega_0/r)\,p = i\rho_0(\Phi-N^2)\,u, \quad (3)$$

where $D_* = d/dr-r^{-1}$ and $\Phi = (\rho_0 r^3)^{-1}D[\rho_0(r^2\Omega_0)^2]$ is the familiar Rayleigh–Synge parameter, which, when positive everywhere in a heterogeneous flow without axial velocity gradients, guarantees stability against axisymmetric modes. After some manipulation these equations can in turn be reduced to the single second-order equation

$$N^2[D(\rho_0 ED^*)-\rho_0 k^2]\,u+[NF+G]\,u = 0, \quad (4)$$

where

$$E = (1+s^2)^{-1}, \quad F = -kD_*\{\rho_0 E[DW_0+sD^*(r\Omega_0)\},$$

$$G = \rho_0 k^2 E[\Phi-2s\Omega_0(DW_0)+s^2 r\Omega_0^2 D(\log\rho_0)].$$

This equation has been given previously by Kurzweg (1969) and represents the governing eigenvalue problem for the inviscid instability of heterogeneous swirling flows with radius-dependent velocity and density when used in conjunction with the boundary conditions $u(R_1) = u(R_2) = 0$.

Equation (4) is reminiscent of the Taylor–Goldstein equation, arising in stability investigations of stratified shearing flows in the presence of a gravitational field, and like this equation is not soluble analytically except for some very special profiles. A numerical solution, such as that given by Hazel (1972) for the Taylor–Goldstein equation, is always possible, however we confine ourselves here to those velocity and density profiles which allow exact solutions in terms of modified Bessel functions. To be able to do this, it is necessary either to avoid the second-order singularity in the equation as ω_i vanishes by having N constant or to require F and G to vanish simultaneously. The former is possible when both Ω_0 and W_0 are constant, the latter for constant density flows when the angular velocity component has the form of a Rankine vortex and the axial velocity does not vary with radius. The density distribution in all our calculations here will be assumed to have a power-law dependence on radius. Clearly, within the framework of these restrictions, we can approximate more complicated profiles only by the standard technique of dividing the flow into subregions inside which the above conditions on the velocity and density hold and then matching the solutions at the interfaces between the subregions using an appropriate set of kinematic and dynamic interfacial conditions. The conditions to be met at each interface are simply that the displacement and total pressure be continuous there. The mathematical forms of these interfacial conditions are readily obtained by integrating (2) and (3) across an interface and letting the integration interval approach zero. This yields, respectively,

$$\langle u/N \rangle = 0, \langle p \rangle - i(u/N)_R[\langle \rho_0 r \Omega_0^2 \rangle] = 0, \tag{5}$$

where $\langle g \rangle = g(R+\epsilon) - g(R-\epsilon)$ represents the difference between conditions on the two sides of the interface at $r = R$ as given by exact solutions of (4) in the adjoining subintervals. When allowance is made for surface-tension effects, the term in the square brackets in the above pressure condition should contain the extra term $-(T/R^2)\,[1-m^2-(kR)^2]$, where T is the surface-tension coefficient. This can be demonstrated by integrating the stability equation after the appropriate surface-tension term has been added. It should be pointed out that Michalke & Timme (1967) used an incorrect interfacial pressure condition in their study of Rankine-vortex instability which does not contain the centrifugal term $\rho_0 r \Omega_0^2$. This implies, as also noted by Uberoi *et al.* (1972), that their calculations for constant density rotating flow with a jump in angular velocity at the interface have to be corrected.

3. Sufficiency condition for stability

Before obtaining some exact solutions of (4), we briefly rederive (see Kurzweg 1969) and then discuss some of the properties of the general sufficiency condition for the stability of the heterogeneous flows under consideration. Such a condition follows directly from (4) by setting $u = N^{\frac{1}{2}}\psi$, multiplying the result by the complex conjugate $\bar{\psi}$, and then integrating over the flow region and using the boundary conditions that ψ vanishes at $r = R_1$ and $r = R_2$. Doing this, one finds the integral expression

$$\int_{R_1}^{R_2} \{-\rho_0 N[E|D^*\psi|^2 + k^2|\psi|^2] + ([F + \tfrac{1}{2}D_*(\rho_0 EDN)] + N^{-1}[G - \tfrac{1}{4}\rho_0 E(DN)^2])\,|\psi|^2\}\, r\,dr = 0, \tag{6}$$

whose imaginary part is

$$2\omega_i \int_{R_2}^{R_2} \left\{\rho_0[E|D^*\psi|^2 + k^2|\psi|^2] + [G - \tfrac{1}{4}\rho_0 E(DN)^2]\frac{|\psi|^2}{|N|^2}\right\} r\,dr = 0. \tag{7}$$

If the term in the second pair of square brackets in (7) remains positive definite throughout the interval $[R_1, R_2]$ then the flow must necessarily remain stable (i.e. $\omega_i = 0$). We thus find that a sufficient condition for stability is

$$[\Phi - \tfrac{1}{4}(DW_0)^2] - 2s(DW_0)[\Omega_0 + \tfrac{1}{4}rD\Omega_0] + s^2[r\Omega_0^2 D(\log\rho_0) - \tfrac{1}{4}r^2(D\Omega_0)^2] \geqslant 0 \tag{8}$$

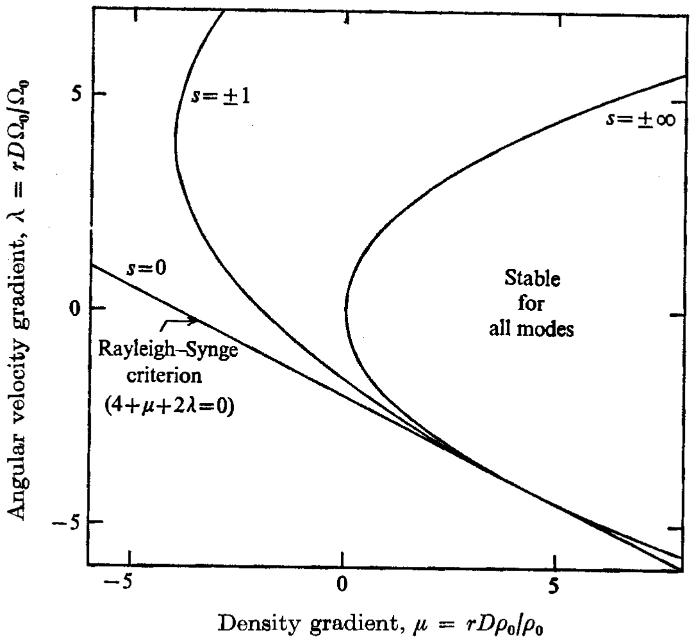

FIGURE 1. Sufficiency condition (9) for the stability of heterogeneous swirling flow for various values of $s = m/kr$ in the absence of an axial velocity gradient.

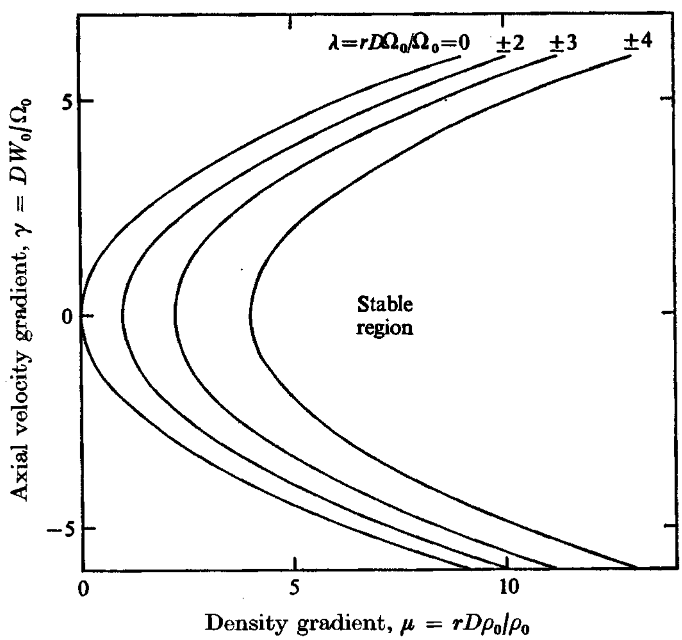

FIGURE 2. Effect of axial velocity gradient on the stability of heterogeneous swirling flow for the potentially least stable mode at $s = (4+\lambda)/\gamma$. Sufficiency condition: $\gamma^2 + \lambda^2 \leqslant 4\mu$.

for all r in $[R_1, R_2]$. This criterion may also be rewritten in the convenient non-dimensional form

$$\Gamma = \frac{(1+s^2)\,\Phi}{[DW_0+s(rD\Omega_0+4\Omega_0)]^2} \geqslant \tfrac{1}{4}, \tag{9}$$

which is reminiscent of the Richardson number criterion encountered in the stability of stratified shearing flows. It should be noted that violations of inequalities (8) and (9) do not necessarily imply instability. On the other hand, as will be supported by the exact solutions given below, the true stability boundary is usually not far removed from that found by setting the left-hand side of inequality (8) to zero. Indeed, for certain flows and mode asymmetry the above sufficiency condition is found to be exact.

Several observations concerning flow stability follow at once from sufficiency condition (9). We note that the flow is stable for all $s = m/rk$ when the Rayleigh–Synge parameter Φ is positive everywhere throughout the flow and at the same time the radial gradients in the angular and axial velocity components remain small. Also, it is observed that swirling flows with constant density, such as those investigated by Howard & Gupta (1962), always violate condition (8) for large enough s, preventing one from obtaining a general sufficiency condition in that limit. Mathematically, this violation stems from the fact that the term G in (4) vanishes for constant density swirling flows subject to azimuthally symmetric disturbances. In view of this it is also not possible to obtain a sufficiency condition for non-rotating flows as there G will vanish for arbitrary s and hence the second-order singularity in the equation for neutral stability does not appear. In figures 1 and 2 we have plotted the stability effects of the non-dimensional density gradient $\mu = rD\rho_0/\rho_0$ as a function of the non-dimensional axial velocity gradient $\gamma = DW_0/\Omega_0$ and angular velocity gradient $\lambda = rD\Omega_0/\Omega_0$ as predicted by inequality (8). It may be seen that the sufficiency condition reduces to the Rayleigh–Synge criterion for axisymmetric modes provided that γ vanishes. However, flow stability for non-axisymmetric modes with a large symmetry parameter $s = m/rk$ can generally be guaranteed only for positive density stratification with small enough gradients in the axial and angular velocity components. The parabolic boundaries shown in figure 2 correspond to the potentially least stable modes at $s = (4+\lambda)/\gamma$, for which Γ, given by (9), has a minimum. Note that the most likely mode of instability for solid-body rotation ($\lambda = 0$) in the absence of an axial velocity gradient ($\gamma = 0$) is predicted to be azimuthally symmetric ($s = \infty$). An increase in either γ or λ reduces the extent of the stable flow region.

An indication of the stability mechanisms responsible for flow instability is given directly by inequality (9). It shows, in analogy with the Richardson number criterion for stratified shearing flows, that the heterogeneous swirling flows considered here are subject both to centrifugal instabilities, occurring when the Rayleigh–Synge parameter is negative somewhere in the flow, and to shear instabilities, which can occur even when Φ is positive. The latter instability is similar to that found in stably stratified shearing flows in a gravitational field. The existence of shear-induced instabilities under conditions where the Rayleigh–Synge criterion is satisfied has recently been supported by some exact solutions for a particular type of swirling flow with radius dependent density (Fung & Kurzweg 1973). The proof by Chandrasekhar (1961, p. 277) that the sign of Φ is the sole determinant for swirling-flow instability would appear to be incorrect in view of criterion (9) and the counter examples to be presented below. It would be equally incorrect to assume that a stratified shearing flow in a gravitational field will always be stable when the density decreases monotonically upwards.

4. Some exact solutions of the stability equation

To obtain both a measure of the validity of our sufficiency condition and some information on the instability growth rates as functions of μ, λ and γ for specific unstable flows, we next obtain explicit solutions of (4) for several velocity and density profiles. To keep within the restrictions discussed earlier, we consider only those flows for which exact solutions in terms of hyperbolic Bessel functions are possible. That is, we restrict ourselves to flows with a step-function or Rankine-vortex distribution for the angular velocity component and a power-law distribution for the density. One of the simplest such flows to analyse is the swirling flow

$$\Omega_0(r) = \Omega_1, \quad W_0(r) = W_1, \quad \rho_0(r) = \rho_1 r^\sigma, \tag{10}$$

where Ω_1, W_1, ρ_1 and σ are constants. For this flow we find that the radial velocity perturbation is given by

$$u_1 = N_1 r^{-\frac{1}{2}\sigma-1}\left\{Ak^{-\nu}\left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{qkrI'_\nu(kqr)}{I_\nu(kqr)}\right]I_\nu(qkr) + Bk^\nu\left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{qkrK'_\nu(kqr)}{K_\nu(kqr)}\right]K_\nu(qkr)\right\}, \tag{11}$$

with the corresponding pressure perturbation given by

$$p_1 = -i\Omega_1^2\rho_1(r/R)^\sigma(n_1^2-4-\sigma)\{r^{-\frac{1}{2}\sigma}[Ak^{-\nu}I_\nu(qkr)+Bk^\nu K_\nu(qkr)]\}. \tag{12}$$

Here, $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions of the first and second kind of order ν, a prime denotes the total derivative with respect to the argument shown and

$$q = (1-(4+\sigma)/n_1^2)^{\frac{1}{2}}, \quad n_1 = k(W_1/\Omega_1)+m-\omega/\Omega_1 = N_1/\Omega_1,$$
$$\nu = (m^2q^2+(2m/n_1+\tfrac{1}{2}\sigma)^2)^{\frac{1}{2}}.$$

The constants A and B can be eliminated by applying boundary conditions $u(R_1) = u(R_2) = 0$, and this leads to the secular relation

$$\begin{vmatrix} \left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa_1 I'_\nu(q\kappa_1)}{I_\nu(q\kappa_1)}\right]I_\nu(q\kappa_1) & \left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa_1 K'_\nu(q\kappa_1)}{K_\nu(q\kappa_1)}\right]K_\nu(q\kappa_1) \\ \left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa_2 I'_\nu(q\kappa_2)}{I_\nu(q\kappa_2)}\right]I_\nu(q\kappa_2) & \left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa_2 K'_\nu(q\kappa_2)}{K_\nu(q\kappa_2)}\right]K_\nu(q\kappa_2) \end{vmatrix} = 0 \tag{13}$$

for the complex oscillation-amplification factor $\omega = \omega_r + i\omega_i$. Here $\kappa_1 = kR_1$ and $\kappa_2 = kR_2$. This determinant is relatively easy to evaluate for axisymmetric disturbances ($s = 0$) and for nearly azimuthally symmetric modes ($s \gg 1$).

We do this by using, respectively, the asymptotic and small-argument expansions for the Bessel functions and find

$$\frac{\omega}{\Omega_1} = k\frac{W_1}{\Omega_1} \pm \left(\frac{\sigma+4}{1+[a\pi/(\kappa_2-\kappa_1)]^2}\right)^{\frac{1}{2}} \tag{14}$$

for $1 \ll kR_1$ and

$$\frac{\omega}{\Omega_1} = k\frac{W_1}{\Omega_1} + m + \frac{m\{\sigma \pm [\sigma(\sigma+h)]^{\frac{1}{2}}\}}{h} \tag{15}$$

for $1 \gg kR_1$. Here $h = m^2 + (\frac{1}{2}\sigma)^2 + [a\pi/\log(R_2/R_1)]^2$, $a = 0, 1, 2, 3, \ldots$.

Equations (14) and (15) clearly show that, for this swirling flow with constant velocity components, axisymmetric modes of instability will occur only when $\sigma < -4$ while azimuthally symmetric modes are expected when $\sigma < 0$. This result is in exact agreement with the sufficiency condition (8) and the stability behaviour shown in figure 1 along the line $\lambda = 0$. Furthermore, it shows that for this flow non-axisymmetric modes are less stable than their axisymmetric counterparts. The instability growth rates do not depend on the axial velocity component since $\gamma = 0$ in this instance. However, instabilities in the form of azimuthally symmetric modes do occur when $\Phi > 0$. Recent experimental measurements by Johnston (1972) on the stability of heterogeneous flows with nearly solid-body rotation and a negative radial density gradient confirm this point.

As a second specific flow we examine the stability of the two-region heterogeneous distribution

$$\left.\begin{aligned} \Omega_0 &= \Omega_1, \quad W_0 = W_1, \quad \rho_0 = \rho_1(r/R)^\sigma \quad \text{for} \quad R_1 \leqslant r \leqslant R, \\ \Omega_0 &= \Omega_2(R/r)^2, \quad W_0 = W_2, \quad \rho_0 = \rho_2 \quad \text{for} \quad R < r \leqslant R_2, \end{aligned}\right\} \tag{16}$$

where Ω_1, Ω_2, W_1, W_2, ρ_1, ρ_2 and σ are constants. This flow represents a rotating jet core with radius-dependent density surrounded by a potential vortex (Rankine) of constant density and uniform angular velocity. By letting σ, W_2 and R_1 vanish and R_2 become infinite, this flow becomes identical to that considered by Uberoi *et al.* (1972) in their study of wing-tip vortex instability. Generally, this flow will be unstable for some modes via the Kelvin–Helmholtz mechanism because of discontinuities in the velocity components at the interface at $r = R$. The sufficiency condition (8) will always be violated for some s at such discontinuities since either λ or γ becomes infinite there. The stability equation (4) is relatively easy to solve for this two-region flow and the appropriate secular relation can be obtained. We have carried out such a calculation and find, after neglecting surface-tension effects and applying the interfacial conditions (5), that the flow stability is governed by

$$\frac{(n_1^2-4-\sigma)\,[I_\nu(q\kappa)-H_1K_\nu(q\kappa)]}{\left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa I_\nu'(q\kappa)}{I_\nu(qk)}\right]I_\nu(q\kappa)-H_1\left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa K_\nu'(q\kappa)}{K_\nu(q)}\right]K_\nu(q\kappa)}$$
$$-\alpha\beta^2 n_2^2\frac{K_m(\kappa)-H_2I_m(\kappa)}{\kappa K_m'(\kappa)-H_2\kappa I_m'(\kappa)} = \alpha\beta^2-1, \tag{17}$$

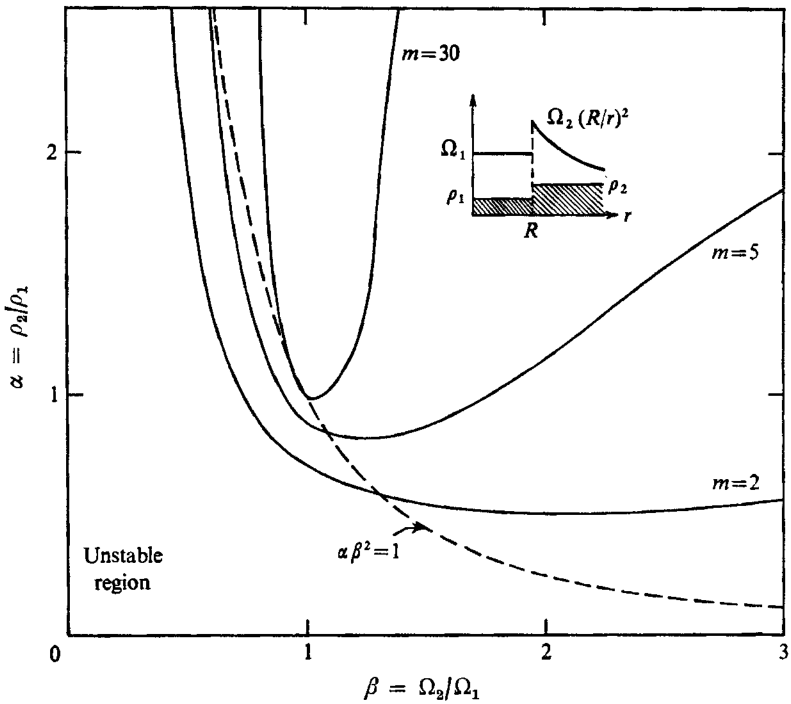

FIGURE 3. Stability boundaries ($\omega_i = 0$) for an unbounded two-region flow for several values of m. – – – –, Rayleigh–Synge criterion for axisymmetric disturbances.

where

$$H_1 = \left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa_1 I_\nu'(q\kappa_1)}{I_\nu(q\kappa_1)}\right] I_\nu(q\kappa_1) \Big/ \left[\frac{2m}{n_1}+\frac{\sigma}{2}+\frac{q\kappa_1 K_\nu'(q\kappa_1)}{K_\nu(q\kappa_1)}\right] K_\nu(q\kappa_1),$$

$$H_2 = K_m'(\kappa_2)/I_m'(\kappa_2),$$

$$\alpha = \rho_2/\rho_1, \quad \beta_2 = \Omega_2/\Omega_1, \quad n_2 = (kW_2+m\Omega_2-\omega)/\Omega_2,$$

with n_1, q, κ_1, κ_2 and ν defined as above.

Evaluation of (17) for arbitrary m and k is generally not possible without the aid of electronic computation. However, several limiting cases can be discussed analytically using the asymptotic and small-argument expansions of the Bessel functions $I_\nu(z)$ and $K_\nu(z)$ and their derivatives. Consider first the case of axisymmetric modes ($m = 0$) for uniform axial velocity ($W_1 = W_2$). Here the secular relation becomes

$$(\omega - kW_1)^2 = \kappa\Omega_1^2(\alpha\beta^2-1)/(qf_1+\alpha f_2), \tag{18}$$

where f_1 and f_2 are lengthy functions of $q_1\kappa_1$ and $q_2\kappa_2$ whose explicit forms are given by Fung (1974). Following essentially the same argument as that used by Alterman (1961) in a related study, it is found that this flow will be stable (i.e. $\omega_i = 0$) whenever $\alpha\beta^2 \geqslant 1$. This result corresponds precisely to the Rayleigh–Synge criterion for such a discontinuous flow, as can readily be shown by integrating the function $\Phi = D[\rho_0(r^2\Omega_0)^2]/\rho_0 r^3$ across the interface at $r = R$. Unstable axisymmetric modes occur for $\alpha\beta^2 < 1$.

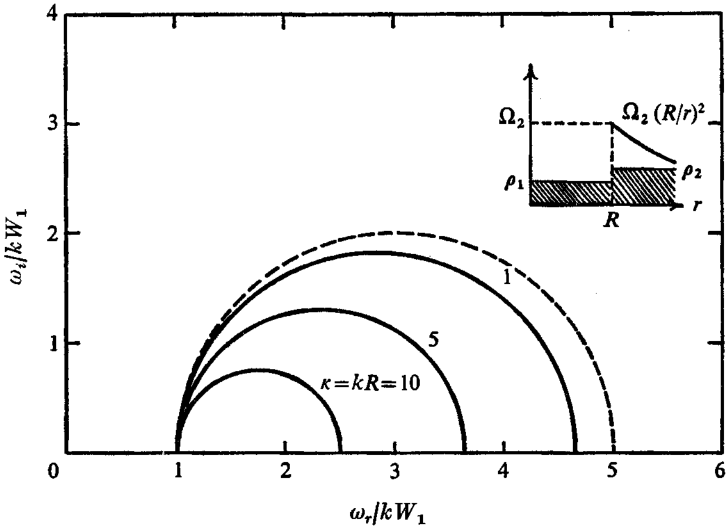

FIGURE 4. Stability characteristics of a non-rotating jet surrounded by a vortex of different density. The special case of axisymmetric modes ($m = 0$) with $kW_1/\Omega_2 = 1$ and $W_2/W_1 = 5$ is shown. – – –, semicircle bound for unstable modes. Stability when

$$\alpha \geqslant \frac{[k(W_2 - W_1)/\Omega_2]^2 + \kappa K_0'(\kappa)/K_0(\kappa)}{\kappa I_0'(\kappa)/I_0(\kappa)}.$$

A second limiting form of the secular relation (17) follows upon setting $k = 0$ and $\sigma = 0$. For such azimuthally symmetric modes the small-argument expansion of the modified Bessel functions yields the expression

$$n_1(n_1 j_1 - 2) + \alpha\beta^2 n_2^2 j_2 = m(\alpha\beta^2 - 1), \tag{19}$$

where $j_i = (-1)^i (R_i^{2m} + R^{2m})/(R_i^{2m} - R^{2m})$ and n_i, α and β are as defined earlier. A neutral-stability boundary occurs when the discriminant of this quadratic in ω vanishes. We have evaluated these stability boundaries for the case of an unbounded flow with $R_1 = 0$ and $R_2 = \infty$ for several different integer values of m. The results are illustrated in figure 3 together with a dashed curve representing the Rayleigh–Synge criterion for axisymmetric modes. Note that the $m \neq 0$ modes are in most instances less stable than their axisymmetric counterpart. This is especially so for large m and, indeed, as m approaches infinity only the special case $\beta = 1$, $\alpha \geqslant 1$ remains stable. Similar behaviour was found by Fung & Kurzweg (1973) for a related flow. The fact that this flow is unstable for large m except for $\beta = 1$ is also supported by the violation of condition (9) occurring for large $D\Omega_0$.

The final limiting form of relation (17) to be considered is that for a non-rotating jet ($\Omega_1 = 0$) with radius-dependent density surrounded by a potential vortex of different but constant density. If in addition the cylinder walls are moved to zero and infinity but the constant axial velocity component in the outer region remains different from that of the jet, (17) assumes the very simple form

$$\frac{(kW_1 - \omega)^2}{\frac{1}{2}\sigma + \kappa I_\nu'(\kappa)/I_\nu(\kappa)} - \alpha\frac{(kW_2 + m\Omega_2 - \omega)^2}{\kappa K_m'(\kappa)/K_m(\kappa)} = \alpha\Omega_2^2. \tag{20}$$

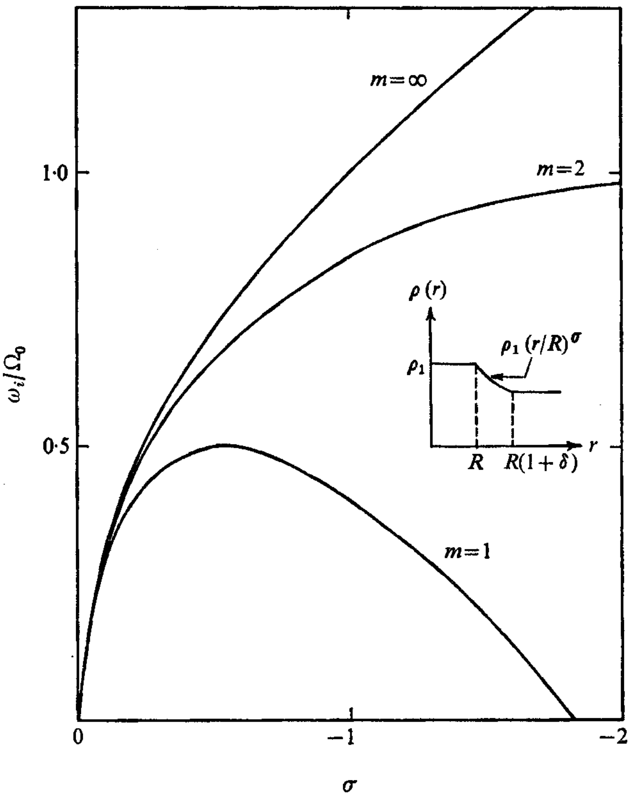

FIGURE 5. Growth rates for azimuthally periodic perturbations in an unbounded three-region swirling flow with constant angular velocity.

For constant density this relation reduces to that which Michalke & Timme (1967) would have obtained if the correct interfacial pressure condition had been used. The values of ω given by this expression are generally complex, implying flow instability. Furthermore, the solutions for axisymmetric modes ($m = 0$) are always found to lie on a semicircle in the complex ω plane. We show this behaviour in figure 4 for the special case $\sigma = 0$, $kW_1/\Omega_2 = 1$ and $W_2/W_1 = 5$ for several different axial wavenumbers k. Note that the instability growth rate for fixed k first increases with increasing α, reaches a maximum at an intermediate value and then decreases with still larger α. For intermediate values of $\kappa = kR$ it is possible to guarantee flow stability when α exceeds the value given in the figure. Another interesting phenomenon seen in figure 4 is that all unstable solutions lie within a semicircle of diameter equal to the range of the axial velocity (Leibovich 1969). This shows very clearly that axisymmetric instabilities can occur even when the Rayleigh–Synge condition ($\alpha\beta^2 \geqslant 1$) is satisfied, provided that axial velocity gradients are taken into consideration. This serves as a counter example to the findings of Chandrasekhar (1961, p. 359).

The last flow whose stability characteristics we wish to examine in some detail is an unbounded three-region flow with arbitrary axial velocity, constant angular velocity and radius-dependent density. We shall examine the stability of this flow only for azimuthally symmetric disturbances as these are expected to be among the least stable of the modes for heterogeneous flows in solid-body rotation. Also, when considering modes with $k = 0$, the possible destabilizing effects of the steady-state axial velocity components disappear. Specifically, we are concerned with the stability of the steady-state distribution

$$\left.\begin{array}{l} \Omega_0 = \text{constant}, \quad W_0 = W_0(r), \\ \rho_0 = \begin{cases} \rho_1 & 0 \leqslant r \leqslant R, \\ \rho_1(r/R)^\sigma & \text{for} \quad R \leqslant r \leqslant R(1+\delta), \\ \rho_1(1+\delta)^\sigma & \text{for} \quad R(1+\delta) \leqslant r < \infty, \end{cases} \end{array}\right\} \tag{21}$$

where ρ_1, σ and δ are constants. This density distribution represents a continuous variation from a constant value ρ_1 for $r \leqslant R$ to a different constant value $\rho_1(1+\delta)^\sigma$ for $r \geqslant R(1+\delta)$. The governing differential equation for this specific flow is again easy to solve in each of the subregions and leads, after application of the interfacial conditions (5) and the boundary conditions that $u(r)$ vanishes at $r = 0$ and $r = \infty$, to the secular relation

$$(2m^2+\sigma\Lambda-2m\nu)-(2m^2+\sigma\Lambda+2m\nu)\,(1+\delta)^{2\nu} = 0, \tag{22}$$

where $\Lambda = 2(1-\omega/m\Omega_0)^{-1}-(1-\omega/m\Omega_0)^{-2}$ and ν, which has been defined previously, becomes $\nu = (m^2+\sigma\Lambda+(\frac{1}{2}\sigma)^2)^{\frac{1}{2}}$. This equation has the obvious solution $\nu = 0$. From it, together with the fact that ω_i will be different from zero only when $\Lambda > 1$, it is seen that the flow is stable when $\sigma > 0$ and unstable when $\sigma < 0$. That is, the flow is stable if the density is an increasing function of radius but unstable when the density is decreasing. This result is consistent with our general sufficiency condition for flow stability (8). In figure 5 we have plotted the effect of increasingly negative σ on the instability growth rate for several different azimuthal wavenumbers and arbitrary δ. Note the monotonic increase in ω_i for large m as σ becomes more negative and the disappearance of the instability as σ approaches zero. Equation (22) yields no complex ω for positive σ, as expected. For negative σ all solutions lie on a semicircle in the complex ω/m plane. The diameter of the semicircle equals Ω_0. This result also holds for arbitrary density distributions, as shown by Fung (1974).

5. Discussion and conclusions

We have found that heterogeneous swirling flows with radius-dependent density are stable to both axisymmetric and non-axisymmetric infinitesimal disturbances whenever the density is an increasing function of radius and at the same time the radial variations of the angular and the axial velocity components remain small. Some exact solutions of the governing stability equations for several different flows confirm this point and support the conclusions drawn from the sufficiency condition obtained earlier. The instability mechanisms present in these swirling flows are two. The first is centrifugal. This form of instability can be suppressed when the product of the density and the square of the circulation is an increasing function of radius (the Rayleigh–Synge criterion). The other is a shear instability mechanism, which becomes important for large radial gradients in the velocity components. Indeed, no degree of stable density stratification can stabilize a flow with a discontinuous velocity profile when surface tension is neglected (Kelvin–Helmholtz instability). It is concluded that the sufficiency condition (9) is a reasonably good indicator of flow stability and that for certain flows and instability modes it predicts the exact location of the stability boundary. The Rayleigh–Synge criterion, a condition for centrifugal stability, is shown by the above calculations to be neither a necessary nor a sufficient condition for flow stability when large shear conditions exist in the flow or non-axisymmetric modes of instability are considered. This fact was first noted for constant density swirling flows by Howard & Gupta (1962). Our results furthermore show that, if one wishes to keep the flow hydrodynamically stable and hence avoid radial mixing of fluid, it is best to require the angular velocity component to remain constant throughout the flow and the fluid to increase in density with increasing radius and to minimize the gradients in the axial velocity. These are precisely the conditions met in centrifuges whose bounding cylinders rotate at the same constant angular velocity but would not be met, for example, by the flow created by tangential injection of fluid into the annular region between stationary cylinders. In the latter case, boundary layers would be formed at the cylinder walls, leading to strong flow instability and subsequent turbulence. The recently commercially introduced density-gradient centrifuges are quite stable to all hydrodynamic modes of instability. In these devices a large positive radial density stratification ($\mu = rD\rho_0/\rho_0 > 0$) is established in a rotating flow of constant angular velocity ($\lambda = rD\Omega_0/\Omega_0 = 0$).

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